Classification of integrable hydrodynamic chains and generating functions of conservation laws

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Abstract

New approach to classification of integrable hydrodynamic chains is established. Generating functions of conservation laws are classified by the method of hydrodynamic reductions. N parametric family of explicit hydrodynamic reductions allows to reconstruct corresponding hydrodynamic chains. Plenty new hydrodynamic chains are found.

Contents

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1 Introduction

The first integrable hydrodynamic chain

$$A_t^k = A_x^{k+1} + kA^{k-1}A_x^0, \qquad k = 0, 1, 2, ... \tag{1}$$

was derived in [1]. The integrability of this hydrodynamic chain was developed in a set of publications ([13], [15], [20], [33]). The “integrability” means the existence of infinitely many conservation laws

$$\partial_t \mathbf{H}_k(A^0, A^1, ..., A^k) = \partial_x G_k(A^0, A^1, ..., A^{k+1}), \qquad k = 0, 1, 2... \tag{2}$$

and infinitely many commuting flows (see [20])

$$A_{t^m}^k = [kA^{k+n-1}\partial_x + n\partial_x A^{k+n-1}]\frac{\partial \mathbf{H}_{m+1}}{\partial A^n}, \qquad k, n, m = 0, 1, 2, ...,$$

where $x \equiv t^0$, $t \equiv t^1$, $\mathbf{H}_0 = A^0$, $\mathbf{H}_1 = A^1$, $\mathbf{H}_2 = A^2 + (A^0)^2$, $\mathbf{H}_3 = A^3 + 3A^0A^1$, ...

The first commuting flow is

$$A_y^k = A_x^{k+2} + A^0A_x^k + (k+1)A^kA_x^0 + kA^{k-1}A_x^1, \qquad k = 0, 1, 2, ..., \tag{3}$$

where $y \equiv t^2$. Eliminating the moments A^1 and A^2 from the equations

$$A_t^0 = A_x^1, \qquad A_t^1 = \partial_x[A^2 + (A^0)^2/2], \qquad A_y^0 = \partial_x[A^2 + (A^0)^2]$$

one can obtain the Khohlov–Zabolotzkaya equation

$$A_{tt}^0 = \partial_x[A_y^0 - A^0A_x^0], \tag{4}$$

which is the dispersionless limit of the KP equation. The compatibility condition $\partial_t(p_y) = \partial_y(p_t)$ of two generating functions of conservation laws for the Benney hydrodynamic chain (1)

$$p_t = \partial_x\left(\frac{p^2}{2} + A^0\right). \tag{5}$$

and for its first commuting flow (3)

$$p_y = \partial_x\left(\frac{p^3}{3} + A^0p + A^1\right) \tag{6}$$

yields (4). Thus, the integrability of this equation is equivalent the integrability of the Benney hydrodynamic chain (see [3], [14], [17], [18]). Substituting the series

$$p = \lambda - \frac{\mathbf{H}_0}{\lambda} - \frac{\mathbf{H}_1}{\lambda^2} - \frac{\mathbf{H}_2}{\lambda^3} - ... \tag{7}$$

in both above equations one can obtain (2) and the infinite series of conservation laws for (3)

$$\partial_y \mathbf{H}_k(A^0, A^1, ..., A^k) = \partial_x Q_k(A^0, A^1, ..., A^{k+2}), \qquad k = 0, 1, 2...$$

Later plenty integrable hydrodynamic chains

$$A_t^k = \sum_{n=0}^{k+1} F_n^k(\mathbf{A}) A_x^n, \qquad k = 0, 1, 2, ..., \qquad \frac{\partial F_n^k}{\partial A^m} = 0, \quad m > k+1 \tag{8}$$

and corresponding 2+1 quasilinear equations (see, for instance, (4)) was found in [10], [11], [9], [12], [19], [23], [32]). Recently some of these integrable hydrodynamic chains were rediscovered (see [2]) and studied in [2], [4], [5], [22], [25], [24]).

At this moment we have several tools allowing to find a complete classification of the integrable hydrodynamic chains (8) and more complicated hydrodynamic chains

$$A_t^k = \sum_{n=0}^{N_k} F_n^k(\mathbf{A}) A_x^n, \qquad k = 0, 1, 2, ..., \qquad \frac{\partial F_n^k}{\partial A^m} = 0, \quad m > M_k, \tag{9}$$

where N_k and M_k are some integers.

1. The existence of an extra commuting flow (an *external* method), successfully used for the Egorov hydrodynamic chains (see [23]);

2. The existence of an extra conservation law (an *external* method), successfully (an incomplete classification) used for the Kupershmidt Poisson bracket and for the (an incomplete classification) Kupershmidt–Manin Poisson bracket (see [19]);

3. Vanishing of the Haantjes tensor (an *internal* method), successfully (unfinished classification) used for the integrable hydrodynamic chains (8) and successfully (complete classification) for the Kupershmidt–Manin Poisson bracket (see [9]);

In this paper we establish the *combined* method based on three key tools, previously successfully utilized in the theory of 2+1 hydrodynamic type systems and 2+1 quasilinear equations (see [10], [11], [12], [32]). These are method of hydrodynamic reductions (see [15], [10]) and the method of pseudopotentials (see [10], [23], [32]) combined with the "concept" of the Gibbons equation (see [27]).

The paper is organized in the following order. In the second section the concept of the Gibbons equation is introduced. In comparison with ([15]) hydrodynamic reductions are described in a conservative form. In the third section a new approach of the classification of integrable hydrodynamic chains via their generating functions of conservation laws is established. In the fourth section explicit hydrodynamic reductions associated with local Hamiltonian structures are discussed. In the fifth section the simple (but nontrivial) generalization of the Benney hydrodynamic chain is investigated. N parametric family of the Hamiltonian hydrodynamic reductions is found. The corresponding Riemann surface is constructed. Thus, N series of conservation laws and commuting flows can be found (see [27]), then infinitely many particular solutions can be obtained by the generalized hodograph method (see [31]). In the sixth section another integrable hydrodynamic chain is described, whose hydrodynamic reductions coincide with hydrodynamic reductions of the hydrodynamic chain described in the fifth section. One particular case of this hydrodynamic chain is connected with 2+1 quasilinear equation determined by the "integrable" Lagrangian considered in [12]. In the seventh section three different approaches allowing to construct commuting flows are presented. The Miura type transformation is used for links between some well-known hydrodynamic chains and corresponding 2+1 quasilinear

systems via generating functions of conservation laws. In the eight section we briefly mention the general case of generating functions of conservation laws connected with arbitrary integrable hydrodynamic chains. This general case is closely connected with the Hamiltonian hydrodynamic chains considered in [28].

2 The Gibbons equation

The Benney hydrodynamic chain (1) is connected with the formal series

$$\lambda = p + \frac{A^0}{p} + \frac{A^1}{p^2} + \frac{A^2}{p^3} + ... \tag{10}$$

by the Gibbons equation (see details below)

$$\lambda_t - p\lambda_x = \frac{\partial \lambda}{\partial p}\left[p_t - \partial_x\left(\frac{p^2}{2} + A^0\right)\right].$$

The *method of hydrodynamic reductions* suggested in [15] (and developed in [10]) means the existence of infinitely many sub-systems (which are called the "hydrodynamic reductions")

$$r_t^i = p^i(\mathbf{r})r_x^i, \qquad i = 1, 2, ..., N, \tag{11}$$

where all moments A^k are functions of N Riemann invariants r^k. This hydrodynamic type system must be consistent with the generating function of conservation laws (5). Then one can obtain

$$\partial_i p = \frac{\partial_i A^0}{p^i - p}. \tag{12}$$

The compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ yield the *system in involution*

$$\partial_i p^k = \frac{\partial_i A^0}{p^i - p^k}, \qquad \partial_{ik} A^0 = 2\frac{\partial_i A^0 \partial_k A^0}{(p^i - p^k)^2}, \qquad i \neq k, \tag{13}$$

which we call the *Gibbons–Tsarev system.*

Any hydrodynamic reduction (11) can be written via first moments A^k ($k = 0, 1, ..., N-1$), where all other moments A^n ($n = N, N+1, ...$) are functions of the first N moments A^k (see details in [15]).

Also, **any** hydrodynamic reduction (11) can be written in the conservative form (see (5))

$$a_t^i = \partial_x\left(\frac{(a^i)^2}{2} + A^0(\mathbf{a})\right), \qquad i = 1, 2, ..., N, \tag{14}$$

where the function A^0 satisfies the Gibbons–Tsarev system (written via field variables a^k)

$$\begin{aligned}(a^i - a^k)\partial_{ik}A^0 \;&=\; \partial_k A^0 \partial_i\left(\sum \partial_n A^0\right) - \partial_i A^0 \partial_k\left(\sum \partial_n A^0\right), \quad i \neq k,\\ (a^i - a^k)\frac{\partial_{ik}A^0}{\partial_i A^0 \partial_k A^0} + (a^k - a^j)&\frac{\partial_{jk}A^0}{\partial_j A^0 \partial_k A^0} + (a^j - a^i)\frac{\partial_{ij}A^0}{\partial_i A^0 \partial_j A^0} = 0, \quad i \neq j \neq k,\end{aligned} \tag{15}$$

which is a consequence of the compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$

$$\frac{p-a^i}{\partial_i A^0}\sum\frac{\partial_{in}A^0}{p-a^n} - \frac{p-a^k}{\partial_k A^0}\sum\frac{\partial_{kn}A^0}{p-a^n} + \frac{(a^i-a^k)\partial_{ik}A^0}{\partial_i A^0 \partial_k A^0}\left(\sum\frac{\partial_n A^0}{p-a^n}-1\right) = 0, \quad (16)$$

where $\partial_i \equiv \partial/\partial a^i$ and (cf. (12))

$$\partial_i p = \frac{\partial_i A^0}{p-a^i}\left(\sum\frac{\partial_n A^0}{p-a^n}-1\right)^{-1}. \quad (17)$$

Example: The simplest hydrodynamic reduction is given by $A^0 = \sum \varepsilon_k a^k$. This is so-called "waterbag" reduction (see [16], [17]). Then (17)

$$\partial_i p = \frac{\varepsilon_i}{p-a^i}\left(\sum\frac{\varepsilon_n}{p-a^n}-1\right)^{-1}$$

can be integrated. The equation of the Riemann surface

$$\lambda = p - \sum \varepsilon_k \ln(p-a^k), \quad (18)$$

where λ is an integration factor, can be expanded (at the infinity $\lambda \to \infty$, $p \to \infty$) in the formal series (10), where $A^k = \Sigma\varepsilon_i(a^i)^{k+1}/(k+1)$, if $\Sigma\varepsilon_i = 0$. If $\Sigma\varepsilon_i \neq 0$, then at first the parameter λ in the equation of the Riemann surface must be re-scaled

$$\lambda - \sum \varepsilon_k \ln\lambda = p - \sum \varepsilon_k \ln(p-a^k).$$

Then all other moments A^k are some non-homogeneous polynomials of $\Sigma\varepsilon_i(a^i)^k$.

This is not an unique choice. For instance, any hydrodynamic reductions can be written in a conservative form in infinitely many ways. Let us mention just two simplest choices here (all other are discussed in [27]):

$$\mathbf{1.} \quad a_t^i = \partial_x\left(\frac{(a^i)^2}{2} + A^0(\mathbf{a},\mathbf{b})\right), \qquad b_t^i = \partial_x(a^i b^i) \qquad i = 1,2,...,N, \quad (19)$$

$$\mathbf{2.} \quad a_t^i = \partial_x\left(\frac{(a^i)^2}{2} + A^0\right), \qquad A_t^0 = \partial_x A^1(\mathbf{a}) \qquad i = 1,2,...,N,$$

For the first choice, the simplest hydrodynamic reduction is given by $A^0 = \Sigma b^k$; for the second choice $A^1 = \Sigma b^k$.

The Riemann invariants are most suitable coordinates to prove such general properties like the existence of infinitely many hydrodynamic reductions parameterized by N arbitrary functions of a single variable (see [15]), but they are very inconvenient for a search of particular and explicit hydrodynamic reductions. Vice versa, it is very difficult to derive the Gibbons–Tsarev system in the coordinates a^k (see (17)), but explicit particular hydrodynamic reductions can be found naturally in many cases (see a lot of examples in [25]).

The *phenomenological* algebro-geometric approach for integrability of symmetric hydrodynamic type systems

$$a_t^i = \partial_x \psi(a^1, a^2, ..., a^N; p)|_{p=a^i} \tag{20}$$

was formulated in [27].

Statement 1: *If the symmetric hydrodynamic type system* (20) *is integrable, then this system has the generating function of conservation laws*

$$p_t = \partial_x \psi(a^1, a^2, ..., a^N; p). \tag{21}$$

Statement 2: *Then some function* $\lambda(a^1, a^2, ..., a^N; p)$ *satisfies the Gibbons equation*

$$\lambda_t - \frac{\partial \psi}{\partial p}\lambda_x = \frac{\partial \lambda}{\partial p}[p_t - \partial_x \psi(\mathbf{a}; p)]. \tag{22}$$

We call the function $\lambda(a^1, a^2, ..., a^N; p)$ the equation of the Riemann surface. This function is a solution of the set of *linear* PDE's

$$A_i^k \frac{\partial \lambda}{\partial u^k} + \frac{\partial \psi}{\partial u^i}\frac{\partial \lambda}{\partial p} = 0,$$

where the matrix A_i^k is given by

$$A_i^k(\mathbf{u}; p) = \left(\frac{\partial \psi}{\partial p}|_{p=u^i} - \frac{\partial \psi}{\partial p}\right)\delta_i^k + \frac{\partial \psi}{\partial u^i}|_{p=u^k}.$$

The Gibbons equation (22) describes a deformation of the Riemann surface $\lambda(\mathbf{a}; p)$. This equation has three distinguish features:

1. if one fixes $\lambda = \text{const}$ (free parameter), then one obtains (21),

2. if one fixes $p = \text{const}$ (free parameter), then one obtains the kinetic equation (a collisionless Vlasov equation) written in so-called Lax form

$$\lambda_{t^1} = \{\lambda, \hat{\mathbf{H}}\} = \frac{\partial \lambda}{\partial x}\frac{\partial \hat{\mathbf{H}}}{\partial p} - \frac{\partial \lambda}{\partial p}\frac{\partial \hat{\mathbf{H}}}{\partial x},$$

where $\hat{\mathbf{H}} = \psi(\mathbf{a}; p)$.

3. if one choose coordinates, which are the Riemann invariants determined by the condition $\partial\lambda/\partial p = 0$, then the corresponding hydrodynamic type system (20) can be written in the diagonal form (cf. (11))

$$r_{t^1}^i = \mu^i(\mathbf{r}) r_x^i, \qquad i = 1, 2, ..., N, \tag{23}$$

where the characteristic velocities

$$\mu^i = \frac{\partial \psi}{\partial p}|_{p=p^i}$$

can be found from the algebraic system $\det A_i^k(\mathbf{u}; p) = 0$; then the corresponding values p^i can be expressed via the Riemann invariants r^k. In this algebro-geometric construction

the Riemann invariants are the branch points $r^i = \lambda|_{\partial\lambda/\partial p=0}$ of the Riemann surface (exactly as it is in the Whitham theory, see [7] and [18]).

Example: The dispersionless limit of the vector NLS (see (19) and [33])

$$a_t^i = \partial_x \left(\frac{(a^i)^2}{2} + \sum b^n \right), \qquad b_t^i = \partial_x(a^i b^i), \qquad i = 1, 2, ..., N \tag{24}$$

is the first known hydrodynamic reduction of the Benney hydrodynamic chain (1) determined by the moment decomposition $A^k = \sum (a^i)^k b^i$. In such case the formal series (10) reduces to the equation of the Riemann surface

$$\lambda = p + \sum \frac{b^k}{p - a^k}. \tag{25}$$

The Zakharov reduction (24) written in the Riemann invariants (23) (see (11))

$$r_t^i = p^i(\mathbf{r}) r_x^i, \qquad i = 1, 2, ..., 2N,$$

where the characteristic velocities $p^i(\mathbf{r})$ can be found from ($\partial\lambda/\partial p = 0$)

$$1 = \sum \frac{b^k}{(p - a^k)^2}$$

has infinitely many conservation laws, which can be obtained from (25) with the aid of the Bürmann–Lagrange series (see, for instance, [21] and some details in [27]).

The compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ must be valid *identically* for any symmetric hydrodynamic type system (20), but a whole set (parameterized by N arbitrary functions of a single variable) of such symmetric hydrodynamic type systems (for each fixed function $\psi(\mathbf{a}; p)$) is *described* by these compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$.

Thus, the problem of the description of semi-Hamiltonian symmetric hydrodynamic type system is the problem of the classification of integrable hydrodynamic chains.

3 The Gibbons–Tsarev system

The **main claim** of this paper is that *the classification of integrable hydrodynamic chains* (9) *is equivalent to the classification of the generating functions of conservation laws* (21) *by the method of hydrodynamic reductions (see* [15], [10]).

In this section we derive integrability conditions, which are a nonlinear PDE's system in involution. This *extended Gibbons–Tsarev system* generalizes the Gibbons–Tsarev system obtained in [15] and describes a *set* of generating functions of conservation laws together with their hydrodynamic reductions parameterized by N arbitrary functions of a single variable.

The simplest case is

$$p_t = \partial_x \psi(u; p). \tag{26}$$

Let us first introduce the new notations

$$f_i \equiv f(u, p^i, p) = \frac{\psi_u}{\psi_p|_{p=p^i} - \psi_p}, \quad f_{ik} \equiv f(u, p^i, p^k) = \frac{\psi_u|_{p=p^k}}{\psi_p|_{p=p^i} - \psi_p|_{p=p^k}}, \quad i \neq k, \tag{27}$$

$$\varphi_{ik}(u,p^i,p^k,p)\equiv\frac{f_{ik}\partial_{p^k}f_k-f_{ki}\partial_{p^i}f_i+f_i\partial_p f_k-f_k\partial_p f_i-\partial_u(f_i-f_k)}{f_i-f_k},\qquad i\neq k. \tag{28}$$

Differentiating the generating function of conservation laws (**26**) with respect to the Riemann invariants r^i (see (**23**)), one obtains

$$\partial_i p=f_i\partial_i u. \tag{29}$$

If $p=p^k$ $(k\neq i)$, then (**29**) reduces to

$$\partial_i p^k=f_{ik}\partial_i u. \tag{30}$$

The compatibility condition $\partial_k(\partial_i p)=\partial_i(\partial_k p)$ yields

$$\partial_{ik}u=\varphi_{ik}\partial_i u\partial_k u. \tag{31}$$

Thus, the nonlinear PDE's system (**30**), (**31**) is in involution iff the functions φ_{ik} *do **not** depend on* p and the compatibility conditions $\partial_j(\partial_i p^k)=\partial_i(\partial_j p^k)$ and $\partial_j(\partial_{ik}u)=\partial_i(\partial_{jk}u)$ are fulfilled *identically*. We call the system (**30**), (**31**) the Gibbons–Tsarev system (cf. (**13**)).

However, in general case the functions φ_{ik} *depend on* p. The compatibility conditions $\partial_k(\partial_i p^j)=\partial_i(\partial_k p^j)$ yields (cf. (**31**))

$$\partial_{ik}u=\bar{\varphi}_{ik}\partial_i u\partial_k u, \tag{32}$$

where (cf. (**28**))

$$\bar{\varphi}_{ik}(u,\mathbf{p})\equiv\varphi_{ik}(u,p^i,p^k,p)|_{p=p^j},\qquad i\neq j\neq k.$$

Thus, the Gibbons–Tsarev system is determined by (**30**) and (**32**).

Finally, one must check the compatibility conditions $\partial_j(\partial_{ik}u)=\partial_i(\partial_{jk}u)$

$$\partial_u(\varphi_{ik}-\varphi_{jk})+\varphi_{ij}(\varphi_{ik}-\varphi_{jk})+f_{ji}\partial_{p^i}\varphi_{ik}+f_{jk}\partial_{p^k}\varphi_{ik}+f_j\partial_p\varphi_{ik}=f_{ij}\partial_{p^j}\varphi_{jk}+f_{ik}\partial_{p^k}\varphi_{jk}+f_i\partial_p\varphi_{jk}$$

following from (**31**), and the compatibility conditions $\partial_j(\partial_{ik}u)=\partial_i(\partial_{jk}u)$ following from (**32**). Such over-determined nonlinear PDE system is said to be the *extended* Gibbons–Tsarev system.

Let me emphasize that the extended Gibbons–Tsarev system is a system on the sole function $\psi(u;p)$ only. *The general solution of this system yields a classification of integrable hydrodynamic chains.*

Remark: Any N hydrodynamic reduction (**23**) can be written in the conservative form (see (**26**))

$$a_t^i=\partial_x\psi(u(\mathbf{a});a^i),\qquad i=1,2,...,N. \tag{33}$$

If the function $\psi(u(\mathbf{a});p)$ determines the integrable hydrodynamic type system (**33**), then the compatibility conditions $\partial_i(\partial_k p)=\partial_k(\partial_i p)$ satisfy *identically*, where $\partial_k\equiv\partial/\partial a^k$ and (cf. (**17**))

$$\partial_i p=\frac{\psi_u\partial_i u}{\psi_p|_{p=a^i}-\psi_p}\left[1+\sum\frac{\psi_u|_{p=a^k}\partial_k u}{\psi_p|_{p=a^k}-\psi_p}\right]^{-1}. \tag{34}$$

If the function $\psi(u;p)$ determines the integrable hydrodynamic chain, then the compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ describe its N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable. If the function $\psi(u;p)$ is unknown, then the compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ and the compatibility conditions $\partial_j(\partial_{ik} u) = \partial_i(\partial_{jk} u)$ yield the extended Gibbons–Tsarev system.

Let us finally emphasize: the Gibbons–Tsarev system describes N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable for any a priori given function $\psi(\mathbf{u};p)$; the extended Gibbons–Tsarev system describes all possible functions $\psi(\mathbf{u};p)$ and their N component hydrodynamic reductions parameterized by N arbitrary functions of a single variable.

4 Explicit Hamiltonian hydrodynamic reductions

Suppose the hydrodynamic reductions (**14**) of the Benney hydrodynamic chain (**1**) (simultaneously, they are hydrodynamic reductions of the Khohlov–Zabolotzkaya equation (**4**)) are Hamiltonian (see details in [**8**])

$$a_t^i = \frac{1}{\varepsilon_i}\partial_x \frac{\partial \mathbf{h}}{\partial a^i},$$

where ε_i are arbitrary constants. Then the Hamiltonian density $\mathbf{h} = \Sigma\varepsilon_k(a^k)^3/6 + f(\Delta)$, where $\Delta = \Sigma\varepsilon_k a^k$ and $f'(\Delta) = A^0(\Delta)$. Substitution $A^0(\Delta)$ in (**16**) yields the choice $A^0(\Delta) = \Delta$ only (up to an insufficient constant factor). This is exactly so-called "waterbag" hydrodynamic reduction (see (**18**)).

Thus, the **main claim** of this section is that the Hamiltonian hydrodynamic reductions (cf. (**33**))

$$a_t^i = \partial_x \psi(\mathbf{u}; a^i)$$

for any hydrodynamic chain determined by the the generating function of conservation laws (see (**22**) and (**26**))

$$p_t = \partial_x \psi(\mathbf{u}; p),$$

one can seek in the form

$$a_t^i = \partial_x \left(\delta_i \frac{\partial \mathbf{h}}{\partial a^i} + \gamma_i \sum_{k\neq i} \gamma_k \frac{\partial \mathbf{h}}{\partial a^k} \right), \tag{35}$$

where δ_i and γ_i are some constants (see examples in [**27**]).

The second step is a reconstruction of the Riemann surface determined by the equation $\lambda(\mathbf{u};p)$ by virtue of (**35**) and a computation of the moments $A^k = \Sigma f_n^k(a^n)$, where the functions $f_n^k(a^n)$ from the expansion of the equation of the Riemann surface at infinity ($\lambda \to \infty, p \to \infty$). Couple of examples are considered below.

5 Generalized Benney hydrodynamic chain

In this section we restrict our consideration on the first simple and very important particular case (cf. (**5**))

$$p_t = \partial_x [U(p) + u]. \tag{36}$$

Theorem: *N component hydrodynamic reductions* (**23**)

$$r_t^i = U'(p^i) r_x^i$$

are described by solutions of the Gibbons–Tsarev system (cf. (**13**))

$$\begin{aligned} \partial_{ik} u &= \frac{2\alpha U'(p^i)U'(p^k) + \beta[U'(p^i) + U'(p^k)] + 2\gamma}{(U'(p^i) - U'(p^k))^2} \partial_i u \partial_k u, \\ \partial_i p &= \frac{\partial_i u}{U'(p^i) - U'(p)}, \qquad \partial_i p^k = \frac{\partial_i u}{U'(p^i) - U'(p^k)}, \end{aligned} \tag{37}$$

where the function $U(p)$ is a solution of the second order ODE with respect to the independent variable p

$$U''(p) = \alpha U'^2(p) + \beta U'(p) + \gamma. \tag{38}$$

The compatibility conditions $\partial_j(\partial_{ik} A^0) = \partial_i(\partial_{jk} A^0)$ are identically satisfied for any constants α, β, γ. In general case ($\alpha \neq 0$) the function $U(p)$ can be written in the parametric form only

$$U = \frac{1}{\alpha(q_1 - q_2)}[q_1 \ln(q - q_1) - q_2 \ln(q - q_2)], \qquad p = \frac{1}{\alpha(q_1 - q_2)}[\ln(q - q_1) - \ln(q - q_2)].$$

If $\alpha = 0$, but $\beta \neq 0$, then without lost of generality $U = e^p$; if $\beta = 0$, then $U = p^2/2$; if $\alpha \neq 0$, but $\beta = 0$ and $\gamma = 0$, then $U = \ln p$; if $\alpha \neq 0$, but $\beta = 0$, then $U = \ln\sinh p$ (or $U = \ln\cosh p$).

The first step in the reconstruction of integrable hydrodynamic chains is an extracting of *any* explicit hydrodynamic reduction (see (**33**))

$$a_t^i = \partial_x[U(a^i) + u(\mathbf{a})]. \tag{39}$$

Then the compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ yields (cf. (**15**), (**16**)) (**38**), two distinct index relationship

$$[U'(a^i) - U'(a^k)](\partial_{ik} u + \alpha \partial_i u \partial_k u) = \partial_k u \partial_i \left(\sum \partial_n u\right) - \partial_i u \partial_k \left(\sum \partial_n u\right), \tag{40}$$

and three distinct index relationship

$$[U'(a^i) - U'(a^k)]\partial_j u \partial_{ik} u + [U'(a^k) - U'(a^j)]\partial_i u \partial_{jk} u + [U'(a^j) - U'(a^i)]\partial_k u \partial_{ij} u = 0,$$

where (see (**34**))

$$\partial_i p = \frac{\partial_i u}{U'(a^i) - U'(p)} \left(\sum \frac{\partial_k u}{U'(a^k) - U'(p)} + 1\right)^{-1}. \tag{41}$$

Following the previous section we are looking for the Hamiltonian hydrodynamic reductions (**39**)

$$a_t^i = \partial_x[U(a^i) + u(\mathbf{a})] = \frac{1}{\varepsilon_i} \partial_x \frac{\partial \mathbf{h}}{\partial a^i}. \tag{42}$$

Theorem: *The Hamiltonian hydrodynamic reductions* (**42**) *are determined by the Hamiltonian density* $\mathbf{h}=\Sigma\varepsilon_k\int U(a^k)da^k+f(\Delta)$, *where* $\Delta=\Sigma\varepsilon_k a^k$, $f'(\Delta)=u(\Delta)$ *and* $u(\Delta)=\Delta$, *if* $\alpha=0$; $u(\Delta)=\ln(\Delta)$, *if* $\alpha\neq 0$.

Proof: can be obtained by the substitution (**42**) in (**40**).

The equation of the Riemann surface (see (**41**)) can be found in quadratures

$$d\lambda=\frac{e^{\beta p+2\alpha U(p)}}{u'(\Delta)}dp+e^{\beta p+2\alpha U(p)}\sum\frac{\varepsilon_k d(p-a^k)}{U'(a^k)-U'(p)} \tag{43}$$

for this Hamiltonian hydrodynamic reduction.

Let us introduce the moments $A^k=\Sigma\varepsilon_n\int U'^k(a^n)da^n$, then the hydrodynamic type system (**42**) can be rewritten as the Hamiltonian hydrodynamic chain

$$A_t^k=\sum_{n=0}^{k+2}F_n^k(\mathbf{A})A_x^n,\qquad k=0,1,2,...$$

determined by the Hamiltonian density $\mathbf{h}(A^0,A^1)=A^1+f(A^0)$ (where $f(A^0)=(A^0)^2/2$, if $\alpha=0$ and $f(A^0)=A^0\ln A^0$) and by the *Dorfman* Poisson bracket (see [**6**])

$$\{A^0,A^0\}=\sum\varepsilon_n\delta'(x-x'),$$

$$\{A^k,A^n\}=[k(\alpha A^{k+n+1}+\beta A^{k+n}+\gamma A^{k+n-1})\partial_x+n\partial_x(\alpha A^{k+n+1}+\beta A^{k+n}+\gamma A^{k+n-1})]\delta(x-x').$$

For instance, this hydrodynamic chain is

$$A_t^0=\partial_x[\alpha A^2+\beta A^1+\left(\gamma+\sum\varepsilon_n\right)A^0]$$

$$A_t^k=(\alpha A^{k+2}+\beta A^{k+1}+\gamma A^k)_x+k(\alpha A^{k+1}+\beta A^k+\gamma A^{k-1})A_x^0,\qquad k=1,2,...,$$

if the Hamiltonian density is $\mathbf{h}(A^0,A^1)=A^1+(A^0)^2/2$.

6 The Gibbons–Tsarev system and hydrodynamic chains

In this section we consider the second simple and very important particular case

$$p_t=\partial_x[V(p)\upsilon]. \tag{44}$$

Then N component hydrodynamic reductions (**23**)

$$r_t^i=\upsilon V'(p^i)r_x^i$$

are compatible with the above generating function of conservation laws iff (see (**29**))

$$\partial_i p=V(p)\frac{\partial_i\ln\upsilon}{V'(p^i)-V'(p)}.$$

It is easy to see that under re-scaling

$$\frac{dp}{V(p)}=d\tilde{p},\qquad V'(p)=U'(\tilde{p}),$$

one get the main formula from the previous example (see (**37**)), where $u = \ln \upsilon$. The corresponding equation is (see (**38**))

$$VV'' = \alpha V'^2 + \beta V' + \gamma, \tag{45}$$

where

$$V(p) = \exp U(\tilde{p}), \qquad dp = \exp U(\tilde{p}) d\tilde{p}. \tag{46}$$

In general case ($\alpha \neq 0$) the function $V(p)$ can be written in the parametric form only

$$V(p) = (q - q_1)^{\frac{q_1}{\alpha(q_1 - q_2)}} (q - q_2)^{-\frac{q_2}{\alpha(q_1 - q_2)}}, \qquad p = \frac{1}{\alpha} \int (q - q_1)^{\frac{q_1}{\alpha(q_1 - q_2)} - 1} (q - q_2)^{-\frac{q_2}{\alpha(q_1 - q_2)} - 1} dq.$$

The corresponding degenerate cases (see the above example) are $V(p) = p^k$, where k is an arbitrary constant, $V(p) = e^p$, $V(p) = \cosh p$ (or $V(p) = \sinh p$) and two others given in the implicit form

$$p = \int \frac{dV}{\ln V}, \qquad p = \int \frac{dV}{\sqrt{\ln V}}.$$

Since the equations describing hydrodynamic reductions of these both examples (**36**) and (**44**) are equivalent, then one can recalculate any hydrodynamic reduction of the one example to a corresponding hydrodynamic reduction of the second example (the Hamiltonian reductions of the generating function of conservation laws (**44**) are found in [**27**]). The relationship (**46**) is nontrivial. For instance, $U(\tilde{p}) = \tilde{p}^2/2$ for the Benney hydrodynamic chain (see (**5**)), but $V(p)$ cannot be expressed via known elementary or special functions. Nevertheless, hydrodynamic reductions of the Benney hydrodynamic chain can be recalculated. One can compare the formulas (**34**) for (**36**) and (**44**)

$$\begin{aligned} \frac{\partial p}{\partial a^i} &= \frac{V(p)\partial \upsilon / \partial a^i}{V'(a^i) - V'(p)} \left[\upsilon + \sum \frac{V(a^k)\partial \upsilon / \partial a^k}{V'(a^k) - V'(p)} \right]^{-1}, \\ \frac{\partial \tilde{p}}{\partial c^i} &= \frac{\partial u / \partial c^i}{U'(c^i) - U'(p)} \left[1 + \sum \frac{\partial u / \partial c^k}{U'(c^k) - U'(p)} \right]^{-1}. \end{aligned}$$

They are equivalent under the transformation (**46**) and under the same transformation for the field variables $a^k \leftrightarrow c^k$

$$V(a^i) = \exp U(c^i), \qquad da^i = \exp U(c^i) dc^i.$$

The equation of the Riemann surface (**43**) also can be recalculated

$$d\lambda = (V(p))^{2\alpha - 1} \exp[\beta \int \frac{dp}{V(p)}] \left(\frac{\upsilon(\Delta) dp}{\upsilon'(\Delta)} + \sum \frac{\varepsilon_k}{V(a^k)} \frac{V(a^k) dp - V(p) da^k}{V'(a^k) - V'(p)} \right),$$

where $\Delta = \Sigma \varepsilon_k \int da^k / V(a^k)$, $\upsilon(\Delta) = \exp \Delta$, if $\alpha = 0$; $\upsilon(\Delta) = \Delta$, if $\alpha \neq 0$.

Example: *The Lagrangian quasilinear equation associated with an elliptic curve.*

The Lagrangian

$$L = \int z_x z_y z_t dx dy dt$$

creates the Euler–Lagrange equation

$$z_t z_{xy} + z_y z_{xt} + z_x z_{yt} = 0, \tag{47}$$

which is an integrable 2+1 quasilinear equation (see [12]). The pseudopotentials (cf. (4), (5) and (6); see also [10], [32]) are

$$\frac{S_x}{z_x} = \zeta(\sigma), \qquad \frac{S_y}{z_y} = \zeta(\sigma) + \frac{\wp'(\sigma) + \varepsilon}{2\wp(\sigma)}, \qquad \frac{S_t}{z_t} = \zeta(\sigma) + \frac{\wp'(\sigma) - \varepsilon}{2\wp(\sigma)}, \tag{48}$$

where $\zeta(\sigma)$ and $\wp'(\sigma)$ are Weiershtrass elliptic functions ($\zeta'(\sigma) = -\wp(\sigma)$, $\wp'^2(\sigma) = 4\wp^3(\sigma) + \varepsilon^2$). Introducing the new functions $a = z_x$, $b = z_y$, $c = z_t$, (47) can be written in the form

$$a_t = c_x, \quad a_y = b_x, \quad b_t = c_y, \quad ca_y + ba_t + ab_t = 0.$$

Correspondingly, (48) can be written as the couple of generating functions of conservation laws

$$p_y = \partial_x \left[\left(\frac{p}{a} + \frac{\wp'(\sigma) + \varepsilon}{2\wp(\sigma)} \right) b \right], \qquad p_t = \partial_x \left[\left(\frac{p}{a} + \frac{\wp'(\sigma) - \varepsilon}{2\wp(\sigma)} \right) c \right], \tag{49}$$

where $p = S_x$ and σ can be found from $\zeta(\sigma) = p/a$.

The reciprocal transformation (see [29], [30])

$$dz = adx + bdy + cdt, \qquad d\tilde{t} = dt, \qquad d\tilde{y} = dy$$

reduces the above Lagrangian to

$$\tilde{L} = \int \frac{x_{\tilde{y}} x_{\tilde{t}}}{x_z^2} dz d\tilde{y} d\tilde{t}$$

and (49) to the couple of generating functions of conservation laws

$$\tilde{p}_{\tilde{y}} = \partial_z \left[\frac{\wp'(\sigma) + \varepsilon}{2\wp(\sigma)} b \right], \qquad \tilde{p}_{\tilde{t}} = \partial_z \left[\frac{\wp'(\sigma) - \varepsilon}{2\wp(\sigma)} c \right], \tag{50}$$

where $\tilde{p} = \zeta(\sigma)$. However, these generating functions of conservation laws are considered in this section (see (44)). Indeed, the equation (45)

$$V \frac{\partial^2 V}{\partial \tilde{p}^2} = 3 \left(\frac{\partial V}{\partial \tilde{p}} \right)^2 + 9 \frac{\partial V}{\partial \tilde{p}} + 6$$

has the general solution given in the parametric form

$$V(\tilde{p}) = \frac{\wp'(\sigma) \pm \varepsilon}{2\wp(\sigma)}, \qquad \tilde{p} = \zeta(\sigma).$$

7 Commuting flows and 2+1 quasilinear systems

In this section three different approaches in construction of commuting flows for any given hydrodynamic chain are presented.

1. Suppose all generating functions of conservation laws (**26**) are enumerated. Then generating functions of conservation laws for commuting flows one should seek in the forms

$$p_{t^1} = \partial_x \psi_1(u^1; p), \qquad p_{t^2} = \partial_x \psi_2(u^1, u^2; p), \ ... \tag{51}$$

The compatibility conditions $\partial_t(p_{t^1}) = \partial_{t^1}(p_t)$, $\partial_t(p_{t^2}) = \partial_{t^2}(p_t)$, ... allow to reconstruct functions ψ_k in quadratures (the corresponding example is given by the Benney hydrodynamic chain (**1**) and the Khohlov–Zabolotzkaya equation (**4**); see (**5**) and (**6**)).

Example: Let us consider the couple of commuting generating functions of conservation laws (cf. (**50**))

$$p_y = \partial_x [V(p)b], \qquad p_t = \partial_x [W(p)c].$$

The compatibility condition $\partial_t(p_y) = \partial_y(p_t)$ yields (**45**)

$$\begin{aligned} VV'' &= (1 + \frac{\beta}{\alpha}) V'^2 + [\gamma - \delta - 2\frac{\beta\delta}{\alpha}] V' + \frac{\delta}{\alpha}(\beta\delta - \alpha\gamma), \\ WW'' &= (1 + \frac{\delta}{\gamma}) W'^2 + [\alpha - \beta - 2\frac{\beta\delta}{\gamma}] W' + \frac{\beta}{\gamma}(\beta\delta - \alpha\gamma), \end{aligned}$$

where 2+1 quasilinear system is ($\alpha, \beta, \gamma, \delta$ are arbitrary constants)

$$b_t = \alpha b c_x + \beta c b_x, \qquad c_y = \gamma c b_x + \delta b c_x.$$

2. Let us replace $u^1 \to p(\zeta)$ and $t^1 \to \tau(\zeta)$ in the first equation (**51**)

$$\partial_{\tau(\zeta)} p(\lambda) = \partial_x \psi_1(p(\zeta), p(\lambda)). \tag{52}$$

Definition: *The equation* (**52**) *is called the generating function of conservation laws and commuting flows.*

All generating functions of conservation laws (**51**) can be obtained by the expansion of (**52**) in the series according the Bürmann–Lagrange series of $p(\zeta)$, where $\partial_{\tau(\zeta)}$ is the formal series, whose coefficients ∂_{t^k} enumerate different commuting flows.

Example: The Benney hydrodynamic chain has the generating function of conservation laws (**5**) and the generating function of conservation laws and commuting flows is

$$\partial_{\tau(\zeta)} p(\lambda) = \partial_x \ln[p(\lambda) - p(\zeta)], \tag{53}$$

where $A^0_{\tau(\zeta)} = \partial_x p(\zeta)$. Substituting the expansion (**7**) for $p(\lambda)$ in the above formula, one can obtain the generating function of commuting flows in the form

$$A^0_{\tau(\zeta)} = \partial_x p(\zeta), \qquad A^1_{\tau(\zeta)} = \partial_x \left(\frac{(A^0)^2}{2} + p(\zeta) \right), ...$$

Substituting the expansion (**7**) for $p(\lambda)$ in the above formula, one can obtain the generating functions of conservation laws in the form (see (**5**) and (**6**))

$$p_{t^1} = \partial_x \left(\frac{p^2}{2} + A^0 \right), \qquad p_{t^2} = \partial_x \left(\frac{p^3}{3} + A^0 p + A^1 \right), ...$$

3. Suppose for any given function $\psi(u; p)$ (see (**26**)) we already know the corresponding Hamiltonian hydrodynamic reductions

$$a^i_{t^1} = \partial_x \left(\bar{g}^{ik} \frac{\partial \mathbf{h}_1}{\partial a^k} \right),$$

where $\bar{g}^{ik}$ is a constant non-degenerate symmetric matrix. Then one can seek the *higher* conservation law density $\mathbf{h}_2$ for this hydrodynamic type system. This conservation law density determines the higher commuting flow

$$a^i_{t^2} = \partial_x \left(\bar{g}^{ik} \frac{\partial \mathbf{h}_2}{\partial a^k} \right).$$

The generating function of conservation laws for this hydrodynamic type system (as well as for the corresponding hydrodynamic chain) can be found by the replacement $a^i \to p$ (see details in [27]). The generating functions of commuting flows is given by

$$a^i_{\tau(\zeta)} = \partial_x \left(\bar{g}^{ik} \frac{\partial p(\zeta)}{\partial a^k} \right).$$

Let us replace $\partial_{\tau(\zeta)} \to \partial_{t^{-1}}$ and, correspondingly, $p(\zeta) \to A^{-1}$, then the generating function of conservation laws (**53**) can be written in the form

$$\tilde{p}_x = \partial_{t^{-1}}(e^{\tilde{p}} + A^{-1}), \tag{54}$$

where the *generating function of the Miura type transformations* (see [26]) is given by $\tilde{p} = \ln(p - A^{-1})$. This generating function (**54**) determines the continuum limit of the discrete KP hierarchy (see [17])

$$B^k_x = B^{k+1}_z + k B^k B^0_z, \qquad k = 0, 1, 2, ...,$$

where $z \equiv t^{-1}$ and $B^0 \equiv A^{-1}$.

Remark: All moments A^k are connected with the moments B^k by the *Miura type transformations*

$$A^0 = A^0(B^0, B^1), \quad A^1 = A^1(B^0, B^1, B^2), \quad A^2 = A^2(B^0, B^1, B^2, B^3), ...,$$

which can be obtained by a substitution the generating function of the Miura type transformations $p = A^{-1} + \exp \tilde{p}$ in (**10**) and a comparison with (see [17])

$$\lambda = e^{\tilde{p}} + B^0 + B^1 e^{-\tilde{p}} + B^2 e^{-2\tilde{p}} + ...$$

Let us substitute the expansion $\partial_{\tau(\zeta)} = \partial_{t^{-1}} + \zeta\partial_{t^{-2}} + ...$ and $p(\zeta) \to A^{-1} + \zeta A^{-2} + ...$ in (**53**). Then the compatibility condition $\partial_{t^{-2}}(\partial_{t^1} p) = \partial_{t^1}(\partial_{t^{-2}} p)$ yields the 2+1 shallow water system (see [32])

$$\partial_{t^{-2}} A^0 = \partial_x A^{-2}, \qquad \partial_{t^1} A^{-1} = \partial_x \left(\frac{(A^{-1})^2}{2} + A^0 \right), \qquad \partial_{t^1} A^{-2} = \partial_x (A^{-1} A^{-2}), \tag{55}$$

where

$$\partial_{t^{-2}} p = \partial_x \frac{A^{-2}}{A^{-1} - p}.$$

The compatibility condition $\partial_{t^{-2}}(\partial_{t^{-1}} p) = \partial_{t^{-1}}(\partial_{t^{-2}} p)$ yields the famous Boyer–Finley equation

$$\partial_{t^{-1}} A^{-1} = \partial_x \ln A^{-2}, \qquad \partial_{t^{-1}} A^{-2} = \partial_{t^{-2}} A^{-1}. \tag{56}$$

In the both case (**55**) and (**56**) N component hydrodynamic reductions (**11**) and (see (**23**))

$$r^i_{t^{-1}} = \frac{1}{p^i - A^{-1}} r^i_x, \qquad r^i_{t^{-2}} = \frac{A^{-2}}{(p^i - A^{-1})^2} r^i_x$$

are compatible (i.e. $\partial_{t^1}(r^i_{t^{-2}}) = \partial_{t^{-2}}(r^i_{t^1})$; see [31], [10])

$$\frac{\partial_k p^i}{p^k - p^i} = \frac{\partial_k \left(\frac{1}{p^i - A^{-1}} \right)}{\frac{1}{p^k - A^{-1}} - \frac{1}{p^i - A^{-1}}} = \frac{\partial_k \left(\frac{A^{-2}}{(A^{-1} - p^i)^2} \right)}{\frac{A^{-2}}{(A^{-1} - p^k)^2} - \frac{A^{-2}}{(A^{-1} - p^i)^2}},$$

iff the functions p^i and A^0 satisfy to the Gibbons–Tsarev system (**13**), where

$$\partial_i A^{-1} = \frac{\partial_i A^0}{p^i - A^{-1}}, \qquad \partial_i \ln A^{-2} = \frac{\partial_i A^0}{(p^i - A^{-1})^2}.$$

It is not easy to verify, if did not take into account, that all these 2+1 quasilinear systems (**4**), (**55**) and (**56**) are members of the unique Benney hydrodynamic *lattice* (see [25]).

8 General case

The **main statement** of this paper is that all integrable hydrodynamic chains described by the generating functions of conservation laws (**21**) can be split into sub-classes according to the number M of the functions u^m. The first such a case (**26**) was considered in the previous section ($M = 1$). The second case ($M = 2$) is

$$p_t = \partial_x \psi(u, \upsilon; p).$$

Then N component hydrodynamic reductions (**23**) are compatible with this generating function if the compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ are fulfilled, where

$$\partial_i p = \frac{\psi_u \partial_i u + \psi_\upsilon \partial_i \upsilon}{\psi_p|_{p=p^i} - \psi_p}.$$

To complete this computation, we need some relationship between u and υ (except trivial link $\upsilon(u)$). Without lost of generality we can choose these field variables u and υ as a conservation law density and a flux, respectively:

$$u_t = \upsilon_x.$$

Since the hydrodynamic type system (23) has this conservation law, then

$$\partial_i \upsilon = \psi_p|_{p=p^i} \partial_i u,$$

and (cf. (27), (29))

$$\partial_i p = \frac{\psi_u + \psi_\upsilon \psi_p|_{p=p^i}}{\psi_p|_{p=p^i} - \psi_p} \partial_i u.$$

However, in such a case the Gibbons–Tsarev system can be derives from the extended compatibility conditions $\partial_i(\partial_k p) = \partial_k(\partial_i p)$ and $\partial_i(\partial_k \upsilon) = \partial_k(\partial_i \upsilon)$.

9 Conclusion and outlook

In this paper a new look on a classification of the integrable hydrodynamic chains is presented. The main object which should be under an investigation is the generating function of conservation laws

$$p_t = \partial_x \psi(\mathbf{u}; p),$$

where all distinct cases are separated by the number M of independent functions u^m. The simplest case (26) is considered in details. Each such case has infinitely many sub-cases enumerated by the number K of independent functions υ^k in the commuting generating functions of conservation laws

$$p_y = \partial_x \varphi(\boldsymbol{\upsilon}; p).$$

Thus, all integrable hydrodynamic chains can be split into sub-classes by virtue of the two numbers M and K only.

Suppose all these functions $\psi(\mathbf{u}; p)$ are found. Then the Gibbons–Tsarev system describing N component hydrodynamic reductions for every function $\psi(\mathbf{u}; p)$ can be derived automatically in the Riemann invariants and in the field variables a^k (which are conservation law densities) simultaneously. Then corresponding hydrodynamic type systems with a local Hamiltonian structure (as well as with a priori prescribed any nonlocal Hamiltonian structure) can be extracted. The equation of the Riemann surface $\lambda(\mathbf{u}; p)$ can be found in quadratures for the hydrodynamic reductions, whose characteristic velocities are invariant with respect to any Lie group symmetry. Asymptotic of the equation of the Riemann surface at the vicinity of any singular point (usually $\lambda \to \infty, p \to \infty$) yields explicit expressions of corresponding integrable hydrodynamic chains. The "flat" hydrodynamic reductions can be used for a construction of a large class of particular solutions for these hydrodynamic chains and related 2+1 quasilinear equations.

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